

HEAT CONTENT FOR CONVOLUTION SEMIGROUPS

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ABSTRACT. Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ be a Lévy process in \mathbb{R}^d and Ω be an open subset of \mathbb{R}^d with finite Lebesgue measure. In this article we consider the quantity $H(t) = \int_{\Omega} \mathbb{P}_x(X_t \in \Omega^c) dx$ which is called the heat content. We study its asymptotic behaviour as t goes to zero for isotropic Lévy processes under some mild assumptions on the characteristic exponent. We also treat the class of Lévy processes with finite variation in full generality.

1. INTRODUCTION

Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ be a Lévy process in \mathbb{R}^d with the distribution \mathbb{P} such that $X_0 = 0$. We denote by $p_t(dx)$ the distribution of the random variable X_t and we use the standard notation \mathbb{P}_x for the distribution related to the process \mathbf{X} started at $x \in \mathbb{R}^d$. The characteristic exponent $\psi(x)$, $x \in \mathbb{R}^d$, of the process \mathbf{X} is given by the formula

$$(1) \quad \psi(x) = \langle x, Ax \rangle - i\langle x, \gamma \rangle - \int_{\mathbb{R}^d} (e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle \mathbf{1}_{\{\|y\| \leq 1\}}) \nu(dy),$$

where A is a symmetric non-negative definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, that is

$$(2) \quad \nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge \|y\|^2) \nu(dy) < \infty.$$

Let Ω and Ω_0 be two non-empty subsets of \mathbb{R}^d such that Ω is open and its Lebesgue measure $|\Omega|$ is finite. We consider the following quantity associated with the process \mathbf{X} ,

$$H_{\Omega, \Omega_0}(t) = \int_{\Omega} \mathbb{P}_x(X_t \in \Omega_0) dx = \int_{\Omega} \int_{\Omega_0 - x} p_t(dy) dx$$

and we use the notation

$$(3) \quad H_{\Omega}(t) = H_{\Omega, \Omega}(t) \quad \text{and} \quad H(t) = H_{\Omega, \Omega^c}(t).$$

The main goal of the present article is to study the asymptotic behaviour of $H_{\Omega}(t)$ as t goes to zero. We observe that

$$H_{\Omega}(t) = |\Omega| - H(t),$$

and thus it suffices to work with the function $H(t)$. The function $u(t, x) = \int_{\Omega - x} p_t(dy)$ is the weak solution of the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= -\mathcal{L} u(t, x), \quad t > 0, x \in \mathbb{R}^d, \\ u(0, x) &= \mathbf{1}_{\Omega}(x), \end{aligned}$$

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where \mathcal{L} is the infinitesimal generator of the process \mathbf{X} , see [17, Section 31]. Therefore, $H_\Omega(t)$ can be interpreted as the amount of *heat* in Ω if its initial temperature is one whereas the initial temperature of Ω^c is zero. In paper [19], the author calls the quantity $H_\Omega(t)$ *heat content* and we will use the same terminology. There are a lot of articles where bounds and asymptotic behaviour of the heat content related to Brownian motion, either on \mathbb{R}^d or on compact manifolds, were studied, see [19], [21], [22], [20], [18], [23]. Recently Acuña Valverde [2] investigated the heat content for isotropic stable processes in \mathbb{R}^d , see also [1] and [3]. In this paper we study the small time behaviour of the heat content associated with rather general Lévy processes in \mathbb{R}^d .

Before we state our results we recall the notion of perimeter. Following [4, Section 3.3], for any measurable set¹ $\Omega \subset \mathbb{R}^d$ we define its perimeter $\text{Per}(\Omega)$ as

$$(4) \quad \text{Per}(\Omega) = \sup \left\{ \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \text{div} \phi(x) \, dx : \phi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d), \|\phi\|_\infty \leq 1 \right\}.$$

We say that Ω is of finite perimeter if $\text{Per}(\Omega) < \infty$. It was shown in [13, 14, 15] that if Ω is an open set in \mathbb{R}^d with finite Lebesgue measure and of finite perimeter then

$$\text{Per}(\Omega) = \pi^{1/2} \lim_{t \rightarrow 0} t^{-1/2} \int_{\Omega} \int_{\Omega^c} p_t^{(2)}(x, y) \, dy \, dx,$$

where

$$p_t^{(2)}(x, y) = (4\pi t)^{-d/2} e^{-\|x-y\|^2/4t}$$

is the transition density of the Brownian motion B_t in \mathbb{R}^d . We also notice that for a non-empty and open set Ω , $\text{Per}(\Omega) > 0$.

Recall that for the Lévy process \mathbf{X} with the transition probability $p_t(dx)$ and the Lévy measure ν we have

$$\lim_{t \rightarrow 0} t^{-1} p_t(dx) = \nu(dx), \quad \text{vaguely on } \mathbb{R}^d \setminus \{0\}.$$

Therefore, we introduce the perimeter $\text{Per}_{\mathbf{X}}(\Omega)$ related to the process \mathbf{X} setting

$$(5) \quad \text{Per}_{\mathbf{X}}(\Omega) = \int_{\Omega} \int_{\Omega^c - x} \nu(dy) \, dx.$$

For instance, if \mathbf{X} is the isotropic (rotationally invariant) α -stable process, denoted by $S^{(\alpha)} = (S_t^{(\alpha)})_{t \geq 0}$, we obtain the well-known α -perimeter, which for $0 < \alpha < 1$ is given by

$$\text{Per}_{S^{(\alpha)}}(\Omega) = \int_{\Omega} \int_{\Omega^c} \frac{dy \, dx}{\|x - y\|^{d+\alpha}}.$$

It was proved in [9] that if Ω has finite Lebesgue measure and is of finite perimeter then $\text{Per}_{S^{(\alpha)}}(\Omega)$ is finite. In the present paper we prove (see Lemma 1) that for any Lévy process with finite variation, cf. [17, Section 21], and for Ω of finite measure and of finite perimeter $\text{Per}(\Omega) < \infty$ the quantity $\text{Per}_{\mathbf{X}}(\Omega)$ is finite as well.

After Pruitt [16], we consider the following function related to the Lévy process \mathbf{X} . For any $r > 0$,

$$(6) \quad h(r) = \|A\| r^{-2} + r^{-1} \left| \gamma + \int_{\mathbb{R}^d} y (\mathbf{1}_{\|y\| < r} - \mathbf{1}_{\|y\| < 1}) \, \nu(dy) \right| + \int_{\mathbb{R}^d} (1 \wedge \|y\|^2 r^{-2}) \, \nu(dy),$$

where (A, γ, ν) is the triplet from (1) and $\|A\| = \max_{\|x\|=1} \|Ax\|$.

Our first result gives a general upper bound for the heat content related to any Lévy process in \mathbb{R}^d .

¹All sets in the paper are assumed to be Lebesgue measurable.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure $|\Omega|$ and of finite perimeter $\text{Per}(\Omega)$, and set $R = 2|\Omega|/\text{Per}(\Omega)$. Let \mathbf{X} be a Lévy process in \mathbb{R}^d . Then there is a constant $C_1 = C_1(d) > 0$ which does not depend on the set Ω such that*

$$H(t) \leq C_1 t \text{Per}(\Omega) \int_{\frac{R}{2} \wedge h^{-1}(1/t)}^R h(r) dr, \quad t > 0.$$

In Proposition 2 we also prove a similar lower bound for a class of isotropic Lévy processes with characteristic exponent satisfying the so-called upper scaling condition, see [7]. Let us recall that a Lévy process \mathbf{X} is isotropic if the measure $p_t(dx)$ is radial (rotationally invariant) for each $t > 0$, equivalently to the matrix $A = \eta I$ for some $\eta \geq 0$, the Lévy measure ν is rotationally invariant and $\gamma = 0$.

In the next theorem we present the asymptotic behaviour of the heat content under the assumption that the Lévy process \mathbf{X} is isotropic and its characteristic exponent is a regularly varying function at infinity with index greater than one. We say that a function $f(r)$ is regularly varying of index α at infinity, denoted by $f \in \mathcal{R}_\alpha$, if for any $\lambda > 0$,

$$\lim_{r \rightarrow \infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha.$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure $|\Omega|$ and finite perimeter $\text{Per}(\Omega)$. If \mathbf{X} is an isotropic Lévy process in \mathbb{R}^d with the characteristic exponent ψ such that $\psi \in \mathcal{R}_\alpha$, for some $\alpha \in (1, 2]$, then²*

$$\lim_{t \rightarrow 0} \psi^-(1/t) H(t) = \pi^{-1} \Gamma(1 - 1/\alpha) \text{Per}(\Omega).$$

The following theorem deals with Lévy processes with finite variation. Recall that according to [17, Theorem 21.9] a Lévy process \mathbf{X} has finite variation on any interval $(0, t)$ if and only if

$$(7) \quad A = 0 \quad \text{and} \quad \int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty.$$

In this case the characteristic exponent has the following simple form

$$\psi(x) = i\langle x, \gamma_0 \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle x, y \rangle}) \nu(dy),$$

where

$$(8) \quad \gamma_0 = \int_{\|y\| \leq 1} y \nu(dy) - \gamma.$$

We notice that for symmetric Lévy processes with finite variation we have $\int_{\|y\| \leq 1} y \nu(dy) = 0$. Thus, for any symmetric Lévy process with finite variation we have $\gamma_0 = 0$. Moreover, for such processes the related function h defined at (6) is Lebesgue integrable on every bounded interval. As we mentioned before, in front of Lemma 1 the quantity $\text{Per}_{\mathbf{X}}(\Omega)$ is finite in the following theorem. For the definition of a directional derivative we refer the reader to Subsection 2.1.

Theorem 3. *Let \mathbf{X} be a Lévy process in \mathbb{R}^d with finite variation. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure $|\Omega|$ and finite perimeter $\text{Per}(\Omega)$. Then*

$$\lim_{t \rightarrow 0} t^{-1} H(t) = \text{Per}_{\mathbf{X}}(\Omega) + \frac{\|\gamma_0\|}{2} V_{\frac{\gamma_0}{\|\gamma_0\|}}(\Omega) \mathbf{1}_{\mathbb{R}^d \setminus \{0\}}(\gamma_0),$$

where $V_u(\Omega)$ is the directional derivative of the indicator function $\mathbf{1}_\Omega$ in the direction u on the unit sphere in \mathbb{R}^d .

²Here $\psi^- = (\psi^*)^-$ is the generalized left inverse of the non-decreasing function $\psi^*(u) = \sup_{s \in [0, u]} \psi(s)$, see Subsection 2.3.

The rest of the paper is organized as follows. We start with a paragraph which gives the list of examples. In Section 2 we present all the necessary facts and tools that we use in the proofs. Section 3 is devoted to the proofs of the above theorems.

Notation. We write $a \wedge b$ for $\min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Positive constants are denoted by C_1, C_2 etc. If additionally C depends on some M , we write $C = C(M)$. We use the notation $f(x) = O(g(x))$ if there is a constant $C > 0$ such that $f(x) \leq Cg(x)$; $f(x) \asymp g(x)$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$; $f(x) \sim g(x)$ at x_0 if $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$. \mathbb{S}^{d-1} stands for the unit sphere in \mathbb{R}^d and $\sigma(du) = \sigma^{d-1}(du)$ is the surface measure.

1.1. Examples. First we consider the isotropic (rotationally invariant) α -stable process in \mathbb{R}^d . The following example shows that our theorems can be regarded as extensions of the results contained in papers [2] and [1].

Example 1. Let $S^{(\alpha)} = (S_t^{(\alpha)})_{t \geq 0}$ be the isotropic α -stable process in \mathbb{R}^d with $\alpha \in (0, 2)$. We recall that the characteristic exponent of $S^{(\alpha)}$ is $x \mapsto c\|x\|^\alpha$, for some $c > 0$, see [17, Theorem 14.14]. The Lévy measure ν of $S^{(\alpha)}$ has the form

$$\nu(dx) = \frac{c_1 dx}{\|x\|^{d+\alpha}}, \quad \text{for some } c_1 > 0.$$

The related function h defined in (6) turns into $h(r) = c_2/r^\alpha$, for some $c_2 > 0$. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure $|\Omega|$ and finite perimeter $\text{Per}(\Omega)$ and let $R = 2|\Omega|/\text{Per}(\Omega)$. Then, by Theorem 1, for any $\alpha \in (0, 2)$,

$$H(t) \leq C_1 \text{Per}(\Omega) t \int_{\frac{R}{2} \wedge t^{1/\alpha}}^R r^{-\alpha} dr, \quad \text{for all } t > 0,$$

and, by Proposition 2, for $\alpha \in [1, 2)$ and t small enough,

$$H(t) \geq C_2 \text{Per}(\Omega) t \int_{t^{1/\alpha}}^R r^{-\alpha} dr.$$

In particular, for $\alpha = 1$ we get

$$\limsup_{t \rightarrow 0} \frac{H(t)}{t \log(1/t)} \leq C_1 \text{Per}(\Omega) \quad \text{and} \quad \liminf_{t \rightarrow 0} \frac{H(t)}{t \log(1/t)} \geq C_2 \text{Per}(\Omega).$$

For $\alpha \in (1, 2)$, by Theorem 2,

$$\lim_{t \rightarrow 0} t^{-1/\alpha} H(t) = \pi^{-1} \Gamma(1 - 1/\alpha) \text{Per}(\Omega)$$

and for $\alpha \in (0, 1)$, by Theorem 3,

$$\lim_{t \rightarrow 0} t^{-1} H(t) = \text{Per}_{S^{(\alpha)}}(\Omega).$$

Here $\gamma_0 = 0$ according to the comments following equation (8).

Example 2. Let \mathbf{X} be a pure jump (i.e. $A = 0$ and $\gamma = 0$ in (1)) isotropic Lévy process in \mathbb{R}^d such that its Lévy measure ν has the form

$$(9) \quad \nu(dx) = \|x\|^{-d} g(1/\|x\|) dx, \quad \text{for some } g \in \mathcal{R}_\alpha, \quad \alpha \in (0, 2).$$

By [8, Proposition 5.1], we conclude that $\psi \in \mathcal{R}_\alpha$. Hence for such processes, for $1 < \alpha < 2$,

$$\lim_{t \rightarrow 0} \psi^-(1/t) H(t) = \pi^{-1} \Gamma(1 - 1/\alpha) \text{Per}(\Omega),$$

and, for $0 < \alpha < 1$,

$$\lim_{t \rightarrow 0} t^{-1} H(t) = \text{Per}_{\mathbf{X}}(\Omega).$$

Typical examples of isotropic Lévy processes satisfying (9) are

- (i) truncated stable process: $g(r) = r^{-\alpha} \mathbf{1}_{(0,1)}(r)$;
- (ii) tempered stable process: $g(r) = r^{-\alpha} e^{-r}$;
- (iii) isotropic Lamperti stable process: $g(r) = r e^{\delta r} (e^r - 1)^{-\alpha-1}$, $\delta < \alpha + 1$;
- (iv) layered stable process: $g(r) = r^{-\alpha} \mathbf{1}_{(0,1)}(r) + r^{-\alpha_1} \mathbf{1}_{[1,\infty)}(r)$, $\alpha_1 \in (0, 2)$.

Example 3. Let \mathbf{X} be a Lévy process which is the independent sum of the Brownian motion and the isotropic α -stable process in \mathbb{R}^d . Then \mathbf{X} is isotropic and its characteristic exponent is $\psi(x) = \eta \|x\|^2 + c \|x\|^\alpha$, for some $\eta, c > 0$ and $\alpha \in (0, 2)$. Clearly we have $\psi \in \mathcal{R}_2$ and whence, by Theorem 2,

$$\lim_{t \rightarrow 0} t^{-1/2} H(t) = \sqrt{\frac{\eta}{\pi}} \text{Per}(\Omega).$$

Example 4. Take $\alpha \in (0, 2)$ and let \mathbf{X} be a symmetric Lévy process in \mathbb{R} which is the independent sum of the isotropic α -stable process and a Lévy process of which the Lévy measure ν has the form

$$(10) \quad \nu(dx) = \sum_{k=1}^{\infty} 2^{k\alpha/2} (\delta_{2^{-k}}(dx) + \delta_{-2^{-k}}(dx)),$$

where δ_x stands for the Dirac measure at x . According to Subsection 2.3, the characteristic exponent $f(x)$ of the process related to $\nu(dx)$ has the form

$$f(x) = 2 \int_0^\infty (1 - \cos(xu)) \nu(du).$$

The characteristic exponent of the isotropic α -stable process is $x \mapsto c|x|^\alpha$, for some $c > 0$, see [17, Theorem 14.14], and whence, by independence, the characteristic exponent of \mathbf{X} equals to

$$\psi(x) = c|x|^\alpha + f(x), \quad c > 0.$$

Since $1 - \cos(v) \asymp v^2$, for $0 < v < 1$, we have for $x > 0$,

$$(11) \quad Cx^2 \int_0^{1/x} u^2 \nu(du) \leq f(x) \leq 4\nu((1/x, \infty)) + x^2 \int_0^{1/x} u^2 \nu(du),$$

where $\nu((1/x, \infty))$ is the ν -measure of the half-line $(1/x, \infty)$. Using formula (10) we obtain that for $x \geq 1$,

$$\int_0^{1/x} u^2 \nu(du) = \sum_{k \geq \log_2 x} 2^{(\alpha/2-2)k} \asymp 2^{(\alpha/2-2) \log_2 x} = x^{\alpha/2-2}.$$

Similarly we have, for $x > 2$,

$$\nu((1/x, \infty)) = \sum_{1 \leq k < \log_2 x} 2^{\alpha k/2} \leq \sum_{1 \leq k \leq [\log_2 x]} 2^{\alpha k/2} = \frac{1 - 2^{\alpha([\log_2 x] + 1)/2}}{1 - 2^{\alpha/2}} \asymp x^{\alpha/2},$$

where $[x]$ stands for the integer part of x . Hence, by (11), $f(x) \asymp |x|^{\alpha/2}$, for $|x| > 2$. We obtain that $\psi(x) \sim c|x|^\alpha$ at infinity and thus, for $\alpha > 1$,

$$\lim_{t \rightarrow 0} t^{-1/\alpha} H(t) = c^{1/\alpha} \pi^{-1} \Gamma(1 - 1/\alpha) \text{Per}(\Omega)$$

and, for $\alpha < 1$,

$$\lim_{t \rightarrow 0} t^{-1} H(t) = \text{Per}_{\mathbf{X}}(\Omega).$$

The next example shows that in the case when \mathbf{X} is not isotropic then the constant in Theorem 2 may depend on the process.

Example 5. For $\alpha > 1$ and $\ell \in \mathcal{R}_0$ we consider a Lévy process \mathbf{X} in \mathbb{R} with the Lévy measure ν of the form

$$\nu(dx) = (c_1 f(1/x)x^{-1}\mathbf{1}_{\{x>0\}} + c_2 f(1/|x|)|x|^{-1}\mathbf{1}_{\{x<0\}}) dx,$$

where $f(r) = r^\alpha \ell(r)$, for $r \geq 1$ and $f(r) = r^\alpha$ for $r < 1$ and for some constants $c_1, c_2 \geq 0$ such that $c_1 + c_2 > 0$. The corresponding characteristic exponent we call ψ .

Let S be the non-symmetric α -stable distribution in \mathbb{R} with the Lévy measure given by $(c_1 x^{-1-\alpha}\mathbf{1}_{\{x>0\}} + c_2 |x|^{-1-\alpha}\mathbf{1}_{\{x<0\}}) dx$ and with the characteristic exponent $\psi^{(\alpha)}$.

We observe that $f^{-1}(1/t)X_t$ converges in law to S . Indeed, it is enough to prove the convergence of characteristic functions and this holds since we easily get that for any x ,

$$\lim_{t \rightarrow 0} t\psi(xf^{-1}(1/t)) = \psi^{(\alpha)}(x).$$

For $\Omega = (a, b)$ we have

$$H(t) = \int_a^b \mathbb{P}(X_t \leq a - x) dx + \int_a^b \mathbb{P}(X_t \geq b - x) dx.$$

A suitable change of variable in both integrals yields

$$H(t) = \int_0^R \mathbb{P}(|X_t| \geq x) dx.$$

Hence,

$$\lim_{t \rightarrow 0} f^{-1}(1/t)H(t) = \lim_{t \rightarrow 0} \int_0^{(b-a)f^{-1}(1/t)} \mathbb{P}(f^{-1}(1/t)|X_t| > u) du = \int_0^\infty \mathbb{P}(|S| > u) du = \mathbb{E}|S|.$$

2. PRELIMINARIES

In this section we collect all the necessary objects and facts that we use in the course of our study. We start with the short presentation of the geometrical tools.

2.1. Geometrical issues. We refer the reader to [4] for a detailed discussion on functions of bounded variation and related topics.

Let $G \subseteq \mathbb{R}^d$ be an open set and $f : G \rightarrow \mathbb{R}$, $f \in L^1(G)$. The total variation of f in G is defined by

$$V(f, G) = \sup \left\{ \int_G f(x) \operatorname{div} \varphi(x) dx : \varphi \in C_c^1(G, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

The directional derivative of f in G in the direction $u \in \mathbb{S}^{d-1}$ is

$$V_u(f, G) = \sup \left\{ \int_G f(x) \langle \nabla \varphi(x), u \rangle dx : \varphi \in C_c^1(G, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

We notice that $V(\mathbf{1}_\Omega, \mathbb{R}^d)$ is the perimeter $\operatorname{Per}(\Omega)$ of the set Ω , cf. (4). Let $V_u(\Omega)$ stand for the quantity $V_u(\mathbf{1}_\Omega, \mathbb{R}^d)$. We mention that, by [4, Proposition 3.62], for any open Ω with Lipschitz boundary $\partial\Omega$ and finite Hausdorff measure $\sigma(\partial\Omega)$ we have

$$\operatorname{Per}(\Omega) = \sigma(\partial\Omega).$$

For any $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$ we define the covariance function g_Ω of Ω as follows

$$(12) \quad g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \mathbf{1}_\Omega(x - y) dx, \quad y \in \mathbb{R}^d.$$

The next proposition collects all the necessary facts concerning the covariance function following the presentation of [10]. This also reveals the link between directional derivatives and covariance functions.

Proposition 1. [10, Proposition 2, Theorem 13 and Theorem 14] *Let $\Omega \subset \mathbb{R}^d$ have finite measure. Then*

- (i) *For all $y \in \mathbb{R}^d$, $0 \leq g_\Omega(y) \leq g_\Omega(0) = |\Omega|$.*
- (ii) *For all $y \in \mathbb{R}^d$, $g_\Omega(y) = g_\Omega(-y)$.*
- (iii) *g_Ω is uniformly continuous in \mathbb{R}^d and $\lim_{y \rightarrow \infty} g_\Omega(y) = 0$.*

Moreover, if Ω is of finite perimeter $\text{Per}(\Omega) < \infty$ then

- (iv) *the function g_Ω is Lipschitz,*

$$2\|g_\Omega\|_{\text{Lip}} = \sup_{u \in \mathbb{S}^{d-1}} V_u(\Omega) \leq \text{Per}(\Omega)$$

and

$$(13) \quad \lim_{r \rightarrow 0} \frac{g_\Omega(0) - g_\Omega(ru)}{|r|} = \frac{V_u(\Omega)}{2}.$$

- (v) *For all $r > 0$ the limit $\lim_{r \rightarrow 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r}$ exists, is finite and*

$$\text{Per}(\Omega) = \frac{\Gamma((d+1)/2)}{\pi^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} \lim_{r \rightarrow 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r} \sigma(du).$$

In particular, (i) and the fact that g_Ω is Lipschitz imply that there is a constant $C = C(\Omega) > 0$ such that

$$(14) \quad 0 \leq g_\Omega(0) - g_\Omega(y) \leq C(1 \wedge \|y\|).$$

2.2. Regular variation. A function $\ell : [x_0, +\infty) \rightarrow (0, \infty)$, for some $x_0 > 0$, is called slowly varying at infinity if for each $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

We say that $f : [x_0, +\infty) \rightarrow (0, +\infty)$ is regularly varying of index $\alpha \in \mathbb{R}$ at infinity, if $f(x)x^{-\alpha}$ is slowly varying at infinity. The set of regularly varying functions of index α at infinity is denoted by \mathcal{R}_α . In particular, if $f \in \mathcal{R}_\alpha$ then

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha, \quad \lambda > 0.$$

The following property, so-called *Potter bounds*, of regularly varying functions will be very useful, see [6, Theorem 1.5.6]. For every $C > 1$ and $\epsilon > 0$ there is $x_0 = x_0(C, \epsilon) > 0$ such that for all $x, y \geq x_0$

$$(15) \quad \frac{f(x)}{f(y)} \leq C \left((x/y)^{\alpha-\epsilon} \vee (x/y)^{\alpha+\epsilon} \right).$$

2.3. Lévy processes. Throughout the paper \mathbf{X} always denotes a Lévy process, that is a càdlàg stochastic process with stationary and independent increments. The characteristic function of X_t has the form $\mathbb{E}e^{i\langle X_t, \xi \rangle} = e^{-t\psi(\xi)}$, where the characteristic exponent ψ is given by (1) with the corresponding Lévy measure ν , cf. (2).

We recall that \mathbf{X} is isotropic if the measures $p_t(dx)$ are all radial. This is equivalent to the radially of the Lévy measure and the characteristic exponent. For isotropic processes the characteristic exponent has the simpler form

$$\psi(x) = \int_{\mathbb{R}^d} (1 - \cos\langle x, y \rangle) \nu(dy) + \eta\|x\|^2,$$

for some $\eta \geq 0$. We usually abuse notation by setting $\psi(r)$ to be equal to $\psi(x)$ for any $x \in \mathbb{R}^d$ with $\|x\| = r > 0$. Since the function ψ is not necessary monotone, it is more convenient to work with the non-decreasing function ψ^* defined by

$$\psi^*(u) = \sup_{s \in [0, u]} \psi(s), \quad u \geq 0.$$

We denote by ψ^- the generalized inverse of the function ψ^* , that is $\psi^-(u) = \inf\{x \geq 0 : \psi^*(x) \geq u\}$. By [6, Theorem 1.5.3], if $\psi \in \mathcal{R}_\alpha$, for some $\alpha > 0$, then $\psi^* \in \mathcal{R}_\alpha$ and thus $\psi^- \in \mathcal{R}_{1/\alpha}$, which implies that $\lim_{t \rightarrow 0} \psi^-(1/t) = \infty$.

To any Lévy process \mathbf{X} we associate the function h defined at (6). According to [16, Formula (3.2)], there is some positive constant $C = C(d)$ such that for any $r > 0$,

$$(16) \quad \mathbb{P}(\|X_t\| \geq r) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} \|X_s\| \geq r\right) \leq Cth(r).$$

We mention that the function h is decreasing and satisfies the doubling property

$$(17) \quad h(2x) \geq h(x)/4, \quad x > 0.$$

For a symmetric Lévy process \mathbf{X} the function h has the simplified form

$$h(r) = \|A\|r^{-2} + \int_{\mathbb{R}^d} (1 \wedge \|y\|^2 r^{-2}) \nu(dy)$$

and for these processes, see [11, Corollary 1],

$$(18) \quad \frac{1}{2}\psi^*(r^{-1}) \leq h(r) \leq 8(1 + 2d)\psi^*(r^{-1}).$$

In the paper we also deal with Lévy processes which have finite variation on any interval $(0, t)$, for $t > 0$. It holds if and only if condition (7) is satisfied. It turns out that for such processes the quantity $\text{Per}_{\mathbf{X}}(\Omega)$ defined at (5) is finite.

Lemma 1. *Assume that \mathbf{X} has finite variation. Then for any $\Omega \subset \mathbb{R}^d$ of finite measure and finite perimeter $\text{Per}(\Omega) < \infty$ we have $\text{Per}_{\mathbf{X}}(\Omega) < \infty$.*

Proof. Using (14) we can write

$$\begin{aligned} \text{Per}_{\mathbf{X}}(\Omega) &= \int_{\Omega} \int_{\Omega^c - x} \nu(dy) dx = \int \int \mathbf{1}_{\Omega}(x) \mathbf{1}_{\Omega^c}(y + x) \nu(dy) dx \\ &= \int_{\mathbb{R}^d} (g(0) - g(y)) \nu(dy) \leq C \int_{\mathbb{R}^d} (1 \wedge \|y\|) \nu(dy). \end{aligned}$$

Further,

$$\int_{\mathbb{R}^d} (1 \wedge \|y\|) \nu(dy) = \int_{\|y\| < 1} \|y\| \nu(dy) + \int_{\|y\| \geq 1} \nu(dy),$$

where the both integrals on the right hand side are finite due to (7) and (2), and the proof is finished. \square

For the detailed discussion on infinitesimal generators of semigroups related to Lévy processes we refer the reader to [17, Section 31] or [5, Section 3.3]. We recall that the heat semigroup $\{P_t\}_{t \geq 0}$ related to the Lévy process \mathbf{X} is given by

$$P_t f(x) = \int_{\mathbb{R}^d} f(x + y) p_t(dy), \quad f \in C_0(\mathbb{R}^d),$$

where $C_0(\mathbb{R}^d)$ is the set of all continuous functions which vanish at infinity. The generator \mathcal{L} of the process \mathbf{X} is a linear operator defined by

$$(19) \quad \mathcal{L}f(x) = \lim_{t \rightarrow 0} \frac{P_t f(x) - f(x)}{t},$$

with the domain $\text{Dom}(\mathcal{L})$ which is the set of all f such that the right hand side of (19) exists. By [17, Theorem 31.5], we have $C_0^2(\mathbb{R}^d) \subset \text{Dom}(\mathcal{L})$ and for any $f \in C_0^2(\mathbb{R}^d)$ it has the form

$$\begin{aligned} \mathcal{L}f(x) &= \sum_{j,k=1}^d A_{jk} \partial_{jk}^2 f(x) + \langle \gamma, \nabla f(x) \rangle \\ &\quad + \int (f(x+z) - f(x) - \mathbf{1}_{\|z\| < 1} \langle z, \nabla f(x) \rangle) \nu(dz), \end{aligned}$$

where (A, γ, ν) is the triplet from (1). For Lévy processes with finite variation we have the following.

Lemma 2. *Let \mathbf{X}^0 be a Lévy process with finite variation and such that $\gamma_0 = 0$, cf. (8). Let f be a Lipschitz function (with constant L) in \mathbb{R}^d with $\lim_{x \rightarrow \infty} f(x) = 0$. Then f belongs to the domain of the generator \mathcal{L}^0 of the process \mathbf{X}^0 , i.e. $f \in \text{Dom}(\mathcal{L}^0)$, and*

$$\mathcal{L}^0 f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(dy).$$

Proof. We take a function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(0) = 1$, $\|\phi\|_{L^1} = 1$ and $\text{supp}(\phi) \subset [0, 1]$. We set $\phi_\epsilon(x) = \epsilon^{-d} \phi(\epsilon^{-1}x)$. It is well known that then the function $f_\epsilon(x) = \phi_\epsilon * f(x)$ belongs to $C_0^\infty(\mathbb{R}^d)$. Moreover, we have $\lim_{\epsilon \rightarrow 0} \|f_\epsilon - f\|_\infty = 0$. Indeed, for any $\delta > 0$,

$$\begin{aligned} |f_\epsilon(x) - f(x)| &\leq \int_{\|y\| < \delta} |\phi_\epsilon(y)| |f(x-y) - f(x)| dy + \int_{\|y\| \geq \delta} |\phi_\epsilon(y)| |f(x-y) - f(x)| dy \\ &\leq L\delta \int_{\|y\| < \delta} |\phi_\epsilon(y)| dy + 2\|f\|_\infty \int_{\|y\| \geq \delta} |\phi_\epsilon(y)| dy \leq L\delta \|\phi\|_{L^1} + 2\|f\|_\infty \delta, \end{aligned}$$

for ϵ small enough. Taking δ small as well, we get the claim.

Moreover, since $\gamma_0 = 0$,

$$\begin{aligned} \mathcal{L}^0 f_\epsilon(x) &= \langle \gamma, \nabla f_\epsilon(x) \rangle + \int (f_\epsilon(x+z) - f_\epsilon(x) - \mathbf{1}_{\|z\| < 1} \langle z, \nabla f_\epsilon(x) \rangle) \nu(dz) \\ &= \langle \gamma_0, \nabla f_\epsilon(x) \rangle + \int (f_\epsilon(x+z) - f_\epsilon(x)) \nu(dz) \\ &= \int_{\mathbb{R}^d} (f_\epsilon(x+y) - f_\epsilon(x)) \nu(dy) \end{aligned}$$

and we deduce that

$$\lim_{\epsilon \rightarrow 0} \mathcal{L}^0 f_\epsilon(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(dy).$$

Finally, since \mathcal{L}^0 is closed, we get that $f \in \text{Dom}(\mathcal{L}^0)$ and

$$\mathcal{L}^0 f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(dy)$$

which finishes the proof. \square

3. PROOFS

3.1. Proof of Theorem 1. Before we prove Theorem 1 we establish the following auxiliary lemma.

Lemma 3. *Let \mathbf{X} be a Lévy process in \mathbb{R}^d . Then*

(i) *there is a constant $C = C(d) > 0$ such that for any $R > 0$,*

$$(20) \quad \int_0^R \mathbb{P}(\|X_t\| \geq x) dx \leq Ct \int_{h^{-1}(1/t) \wedge \frac{R}{2}}^R h(r) dr, \quad t > 0.$$

(ii) *The related function $H(t)$ introduced in (3) has the following form*

$$(21) \quad H(t) = \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(y)) p_t(dy).$$

Proof. We start with the proof of (i). Using (16) we clearly get that for some $C > 0$ and any $t > 0$,

$$\mathbb{P}(\|X_t\| \geq x) \leq C(th(x) \wedge 1).$$

Observe that $th(x) \geq 1$ if and only if $x \leq h^{-1}(1/t)$ and thus we set $\beta = h^{-1}(1/t)$. For any $R > 0$, we have

$$(22) \quad \int_0^R \mathbb{P}(\|X_t\| \geq x) dx \leq C \left(\int_0^{\beta \wedge \frac{R}{2}} dx + t \int_{\beta \wedge \frac{R}{2}}^R h(x) dx \right).$$

First we consider the case $\beta \leq R/2$, which is equivalent to $t \leq 1/h(R/2)$. We estimate the second integral in (22) as follows

$$t \int_\beta^R h(x) dx \geq t \int_\beta^{2\beta} h(x) dx \geq th(2\beta)\beta \geq \beta/4.$$

In the last inequality we used the doubling property (17). We obtain that

$$\int_0^R \mathbb{P}(\|X_t\| \geq x) dx \leq 5Ct \int_\beta^R h(x) dx$$

as desired. Next, assume that $R/2 < \beta$. By monotonicity of h , we have

$$t \int_{R/2}^R h(x) dx \geq \frac{tR}{2} h(R) \geq \frac{tR}{2} h(2\beta) \geq R/8.$$

This together with (22) imply

$$\int_0^R \mathbb{P}(\|X_t\| \geq x) dx \leq 5Ct \int_{R/2}^R h(x) dx$$

and this gives (i).

For (ii) we write

$$\begin{aligned} H(t) &= \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) (1 - \mathbb{P}(X_t \in \Omega - x)) dx \\ &= |\Omega| - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_\Omega(x) \mathbf{1}_\Omega(x+y) dx p_t(dy) \\ &= g_\Omega(0) - \int_{\mathbb{R}^d} g_\Omega(-y) p_t(dy), \end{aligned}$$

and using symmetry of g_Ω (see (ii) of Proposition 1) we get (21). □

Proof of Theorem 1. We set $R = 2|\Omega|/\text{Per}(\Omega)$ and split the integral in (21) into two parts

$$H(t) = \int_{\|y\|>R} (g_\Omega(0) - g_\Omega(y)) p_t(dy) + \int_{\|y\|\leq R} (g_\Omega(0) - g_\Omega(y)) p_t(dy) = I_1 + I_2.$$

Using (i) of Proposition 1 we estimate I_1 as follows

$$I_1 \leq |\Omega| \mathbb{P}(\|X_t\| > R) \leq \frac{\text{Per}(\Omega)}{2} \int_0^R \mathbb{P}(\|X_t\| > s) ds.$$

Next, by (iv) of Proposition 1 we obtain

$$\begin{aligned} I_2 &\leq \frac{\text{Per}(\Omega)}{2} \int_{\|y\|\leq R} \|y\| p_t(dy) = \frac{\text{Per}(\Omega)}{2} \int_{\|y\|\leq R} \int_0^{\|y\|} ds p_t(dy) \\ &= \frac{\text{Per}(\Omega)}{2} \int_0^R \int_{s<\|y\|\leq R} \|y\| p_t(dy) ds \leq \frac{\text{Per}(\Omega)}{2} \int_0^R \mathbb{P}(\|X_t\| > s) ds. \end{aligned}$$

These estimates together imply

$$H(t) \leq \text{Per}(\Omega) \int_0^R \mathbb{P}(\|X_t\| > s) ds.$$

Hence, applying (i) of Lemma 3 we deduce the result. \square

In the following Proposition 2, we provide a lower bound for the heat content related to an isotropic Lévy process with the characteristic exponent satisfying some scaling condition. We start with a useful lemma.

Lemma 4. *Let \mathbf{X} be an isotropic Lévy process with the radial characteristic exponent ψ . Suppose that there is a constant $C > 0$ such that for some $\alpha \in (0, 2)$,*

$$(23) \quad C^{-1}\psi(x) \leq \psi(y) \leq C \left(\frac{y}{x}\right)^\alpha \psi(x), \quad 1 < x < y.$$

Then there exists $c > 0$ such that

$$\mathbb{P}(\|X_t\| > r) \geq c(1 - e^{th(r)}), \quad t, r < 1.$$

Proof. We observe that the left hand side inequality in (23) implies that

$$(24) \quad \psi(x) \geq C^{-1}\psi^*(x), \quad x > 1.$$

Thus, proceeding exactly in the same fashion as in the proof of [7, Lemma 14], we obtain

$$\mathbb{P}(\|X_t\| > r) \geq C_1(1 - e^{t\psi^*(1/r)}), \quad t, r < 1.$$

Finally, inequality (18) yields

$$\psi^*(r) \geq ch(1/r), \quad r > 0,$$

and the proof is finished. \square

Proposition 2. *Let \mathbf{X} be an isotropic Lévy process in \mathbb{R}^d with the radial characteristic exponent ψ which satisfies condition (23). Assume also that the related function h (see (6)) is not Lebesgue integrable around zero. Then, for any open $\Omega \subset \mathbb{R}^d$ with finite measure and of finite perimeter, there exists $C > 0$ which does not depend on the set Ω such that, for t small enough,*

$$H(t) \geq Ct \text{Per}(\Omega) \int_{h^{-1}(1/t)}^R h(r) dr,$$

where $R = 2|\Omega|/\text{Per}(\Omega)$.

Proof. We first consider the case $d \geq 2$. Since h is not integrable around 0, it is unbounded and so does ψ due to inequality (18). Therefore, \mathbf{X} is not a compound Poisson process. Hence, by [24, (4.6)], all the transition probabilities $p_t(dx)$ are absolutely continuous with respect to the Lebesgue measure. Since p_t are radial, we have $p_t(x) = p_t(\|x\|e_d)$ with $e_d = (0, \dots, 0, 1)$ and as a result by polar coordinates we get that, for any $u_1, u_2 \in [0, +\infty]$,

$$(25) \quad \mathbb{P}(u_1 < \|X_t\| < u_2) = \int_{\mathbb{R}^d} \mathbf{1}_{\{(u_1, u_2)\}}(\|w\|) p_t(w) dw = \sigma(\mathbb{S}^{d-1}) \int_{u_1}^{u_2} r^{d-1} p_t(re_d) dr.$$

Applying (25) in (21) we obtain that for any $\delta > 0$,

$$H(t) \geq \int_0^\delta r^{d-1} p_t(re_d) \int_{\mathbb{S}^{d-1}} (g_\Omega(0) - g_\Omega(ru)) \sigma(du) dr = \int_0^\delta r^d p_t(re_d) \mathcal{M}_\Omega(r) dr,$$

where

$$\mathcal{M}_\Omega(r) = \int_{\mathbb{S}^{d-1}} \frac{g_\Omega(0) - g_\Omega(ru)}{r} \sigma(du).$$

Using (v) of Proposition 1 and applying Fatou's lemma, we get that $\mathcal{M}_\Omega(r) \geq C \text{Per}(\Omega)$, for some positive constant $C = C(d)$ and for r small enough. Hence, for δ small enough,

$$\begin{aligned} H(t) &\geq C \text{Per}(\Omega) \int_0^\delta r^d p_t(re_d) dr = C \text{Per}(\Omega) \int_0^\delta \int_0^r du r^{d-1} p_t(re_d) dr \\ &= \frac{C \text{Per}(\Omega)}{\sigma(\mathbb{S}^{d-1})} \int_0^\delta \int_{u < \|y\| < \delta} p_t(y) dy du \\ &= \frac{C \text{Per}(\Omega)}{\sigma(\mathbb{S}^{d-1})} \int_0^\delta \mathbb{P}(u < \|X_t\| < \delta) du \\ &= \frac{C \text{Per}(\Omega)}{\sigma(\mathbb{S}^{d-1})} \left(\int_0^\delta \mathbb{P}(\|X_t\| > u) du - \delta \mathbb{P}(\|X_t\| > \delta) \right), \end{aligned}$$

where in the second equality we used (25). By Lemma 4, there is a constant $C_1 = C_1(d) > 0$ such that, for u small enough,

$$\mathbb{P}(\|X_t\| > u) \geq C_1 t h(u).$$

This and (16) imply that, for δ small enough,

$$(26) \quad H(t) \geq C_2 t \text{Per}(\Omega) \left(\int_{h^{-1}(1/t)}^\delta h(u) du - C_3 \delta h(\delta) \right).$$

Since ψ is continuous and $\psi(0) = 0$ we have by (18),

$$\lim_{t \rightarrow 0} h^{-1}(1/t) = 0.$$

Further, since h is not integrable around zero, the integral on the right hand side of (26) tends to infinity for any $\delta > 0$, as t goes to zero. This implies that, for t small enough,

$$\begin{aligned} \int_{h^{-1}(1/t)}^\delta h(u) du - C_3 \delta h(\delta) &= \int_{h^{-1}(1/t)}^R h(u) du - \int_\delta^R h(u) du - C_3 \delta h(\delta) \\ &= \int_{h^{-1}(1/t)}^R h(u) du \left(1 - \frac{\int_\delta^R h(u) du + C_3 \delta h(\delta)}{\int_{h^{-1}(1/t)}^R h(u) du} \right) \\ &\geq \frac{1}{2} \int_{h^{-1}(1/t)}^R h(u) du. \end{aligned}$$

Using this and (26) we obtain that there is some $C_4 > 0$ which does not depend on Ω such that, for t small enough,

$$H(t) \geq C_4 t \operatorname{Per}(\Omega) \int_{h^{-1}(1/t)}^R h(u) \, du,$$

and the proof is finished for $d \geq 2$.

At last, in the case $d = 1$ we use (21),

$$H(t) \geq \int_0^\delta \frac{g_\Omega(0) - g_\Omega(x)}{x} x p_t(dx),$$

and application of (v) of Proposition 1 with $d = 1$ gives that, for $0 < x$ small enough,

$$\frac{g_\Omega(0) - g_\Omega(x)}{x} \geq C \operatorname{Per}(\Omega).$$

Thus, for δ small enough, by symmetry of \mathbf{X} ,

$$\begin{aligned} H(t) &\geq C \operatorname{Per}(\Omega) \int_0^\delta x p_t(dx) = C \operatorname{Per}(\Omega) \int_0^\delta \int_0^x du p_t(dx) \\ &= C \operatorname{Per}(\Omega) \int_0^\delta \int_u^\delta p_t(dx) \, du \\ &= \frac{C}{2} \operatorname{Per}(\Omega) \int_0^\delta \mathbb{P}(u < |X_t| < \delta) \, du. \end{aligned}$$

The result is concluded by the same reasoning as for $d \geq 2$. \square

3.2. Proof of Theorem 2. We start with the following auxiliary lemma.

Lemma 5. *Let \mathbf{X} be an isotropic Lévy process in \mathbb{R}^d with the radial transition probability $p_t(dx)$. Assume that its characteristic exponent $\psi \in \mathcal{R}_\alpha$ with $\alpha \in (1, 2]$. Then $p_t(dx) = p_t(x)dx$ and*

$$(27) \quad \lim_{t \rightarrow 0} \frac{p_t\left(\frac{s}{\psi^-(1/t)} e_d\right)}{(\psi^-(1/t))^d} = p_1^{(\alpha)}(se_d),$$

where $p_t^{(\alpha)}(x)$ is the transition density of the isotropic α -stable process in \mathbb{R}^d when $1 < \alpha < 2$ and $p_t^{(2)}(x)$ is the transition density of the Brownian motion in \mathbb{R}^d .

Proof. Since $\psi \in \mathcal{R}_\alpha$, $\alpha \in (1, 2]$, we have

$$\lim_{r \rightarrow \infty} \frac{\psi(r)}{\log(1+r)} = \infty,$$

and this implies that $p_t(dx) = p_t(x)dx$ with the density $p_t \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, see e.g. [12, Theorem 1].

By the Fourier inversion formula, see [5, Section 3.3],

$$(28) \quad \frac{p_t\left(\frac{s}{\psi^-(1/t)} e_d\right)}{(\psi^-(1/t))^d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \cos\langle se_d, \xi \rangle e^{-t\psi(\psi^-(1/t)\xi)} \, d\xi.$$

Since ψ is continuous, $\psi(\psi^-(1/t)) = 1/t$. Hence

$$\frac{\psi(\psi^-(1/t)\xi)}{1/t} = \frac{\psi(\psi^-(1/t)\xi)}{\psi(\psi^-(1/t))} \sim \|\xi\|^\alpha, \quad t \rightarrow 0,$$

and this leads to

$$\lim_{t \rightarrow 0} e^{-t\psi(\psi^-(1/t)\xi)} = e^{-\|\xi\|^\alpha}.$$

Therefore, to finish the proof we apply the Dominated convergence theorem. We split the integral in (28) into two parts. According to the Potter bounds (15) there is $r_0 > 0$ such that, for t small enough and $\|\xi\| \geq r_0$,

$$t\psi(\psi^-(1/t)\xi) = \frac{\psi(\psi^-(1/t)\xi)}{\psi(\psi^-(1/t))} \geq \frac{1}{2}\|\xi\|^{\alpha/2}.$$

This implies that $e^{-t\psi(\psi^-(1/t)\xi)} \leq e^{-\|\xi\|^{\alpha/2}/2}$, for $\|\xi\| \geq r_0$ and t small enough. For $\|\xi\| < r_0$ we bound $e^{-t\psi(\psi^-(1/t)\xi)}$ by one. The Dominated convergence theorem followed by the Fourier inversion formula proves (27). \square

Proof of Theorem 2. By (21),

$$H(t) = \int_{\mathbb{R}^d} p_t(x) (g_\Omega(0) - g_\Omega(x)) dx.$$

Since $g_\Omega(x) \leq g_\Omega(0) = |\Omega|$, $x \in \mathbb{R}^d$ (see Proposition 1 (i)), for any given $\delta > 0$, we can split the integral into two parts

$$(29) \quad \begin{aligned} H(t) &= \int_{\|x\| \leq \delta} p_t(x) (g_\Omega(0) - g_\Omega(x)) dx + \int_{\|x\| > \delta} p_t(x) (g_\Omega(0) - g_\Omega(x)) dx \\ &= I_1 + I_2. \end{aligned}$$

We estimate I_2 using (16),

$$\int_{\|x\| > \delta} p_t(x) (g_\Omega(0) - g_\Omega(x)) dx \leq |\Omega| \mathbb{P}(\|X\|_t > \delta) = O(t).$$

Since $\psi \in \mathcal{R}_\alpha$, $1 < \alpha \leq 2$, [6, Theorem 1.5.12] yields $\psi^- \in \mathcal{R}_{1/\alpha}$ and thus $\psi^-(1/t) I_2 \rightarrow 0$ as t tends to zero. We are left to study the integral I_1 . Recall that the radially of p_t implies that $p_t(x) = p_t(re_d)$, where $\|x\| = r$ and $e_d = (0, \dots, 0, 1)$. Changing variables into polar coordinates we obtain

$$\psi^-(1/t) I_1 = \psi^-(1/t) \int_0^\delta r^{d-1} p_t(re_d) \int_{\mathbb{S}^{d-1}} (g_\Omega(0) - g_\Omega(ru)) \sigma(du) dr.$$

Making substitution $r = s/\psi^-(1/t)$ we get

$$\psi^-(1/t) I_1 = \int_0^{\delta\psi^-(1/t)} s^d \frac{p_t\left(\frac{s}{\psi^-(1/t)}e_d\right)}{(\psi^-(1/t))^d} \mathcal{M}_\Omega(t, s) ds,$$

where

$$\mathcal{M}_\Omega(t, s) = \int_{\mathbb{S}^{d-1}} \frac{g_\Omega(0) - g_\Omega\left(\frac{s}{\psi^-(1/t)}u\right)}{s/\psi^-(1/t)} \sigma(du).$$

We claim that for any fixed $M > 0$,

$$(30) \quad \lim_{t \rightarrow 0} \int_0^M s^d \frac{p_t\left(\frac{s}{\psi^-(1/t)}e_d\right)}{(\psi^-(1/t))^d} \mathcal{M}_\Omega(t, s) ds = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \text{Per}(\Omega) \int_0^M s^d p_1^{(\alpha)}(se_d) ds.$$

To show the claim we use the Dominated convergence theorem. By Proposition 1 (iv-v),

$$0 \leq \mathcal{M}_\Omega(t, s) \leq \frac{1}{2} \text{Per}(\Omega) \sigma(\mathbb{S}^{d-1})$$

and, for any $s > 0$,

$$\lim_{t \rightarrow 0} \mathcal{M}_\Omega(t, s) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \text{Per}(\Omega).$$

Next, by [7, Formula (23)], for $s \leq M$,

$$\frac{p_t\left(\frac{s}{\psi^-(1/t)}e_d\right)}{(\psi^-(1/t))^d} \leq \frac{p_t(0)}{(\psi^-(1/t))^d} \leq C(M)$$

and, by Lemma 5,

$$\lim_{t \rightarrow 0} \frac{p_t\left(\frac{s}{\psi^-(1/t)}e_d\right)}{(\psi^-(1/t))^d} = p_1^{(\alpha)}(se_d).$$

Hence the Dominated convergence theorem implies (30).

Further, we have

$$\begin{aligned} \int_M^{\delta\psi^-(1/t)} s^d \frac{p_t\left(\frac{s}{\psi^-(1/t)}e_d\right)}{(\psi^-(1/t))^d} ds &= \psi^-(1/t) \int_{M/\psi^-(1/t)}^{\delta} s^d p_t(se_d) ds \\ &= \psi^-(1/t) \int_{M/\psi^-(1/t)}^{\delta} \int_0^s du s^{d-1} p_t(se_d) ds = \psi^-(1/t) \int_0^{\delta} \int_{(M/\psi^-(1/t)) \vee u}^{\delta} s^{d-1} p_t(se_d) ds du \\ &\leq \psi^-(1/t) \int_0^{\delta} \mathbb{P}(\|X_t\| > (M/\psi^-(1/t)) \vee u) du \\ &\leq M \mathbb{P}(\|X_t\| > M/\psi^-(1/t)) + \psi^-(1/t) \int_{M/\psi^-(1/t)}^{\delta} \mathbb{P}(\|X_t\| > u) du. \end{aligned}$$

Now, we notice that combining (16) and (18), we obtain $\mathbb{P}(\|X_t\| > r) \leq C t \psi^*(1/r)$. Thus, using Potter bounds with $\epsilon < \alpha - 1$, we estimate, for t small enough, the first term as follows

$$\begin{aligned} M \mathbb{P}(\|X_t\| > M/\psi^-(1/t)) &\leq M t \psi^*(\psi^-(1/t)/M) \\ &\leq C_1 M \frac{\psi^*(\psi^-(1/t)/M)}{\psi^*(\psi^-(1/t))} \leq C_2 M^{1-\alpha+\epsilon}. \end{aligned}$$

We proceed similarly with the second term. Applying Karamata's theorem [6, Proposition 1.5.8] and Potter bounds we obtain that for t small enough

$$\begin{aligned} \psi^-(1/t) \int_{M/\psi^-(1/t)}^{\delta} \mathbb{P}(\|X_t\| > u) du &\leq t \psi^-(1/t) \int_{M/\psi^-(1/t)}^{\delta} \psi^*(u^{-1}) du \\ &\leq C_3 M (\alpha + 1)^{-1} t \psi^*(\psi^-(1/t)/M) \\ &\leq C_4 M (\alpha + 1)^{-1} \frac{\psi^*(\psi^-(1/t)/M)}{\psi^*(\psi^-(1/t))} \leq C_5 (\alpha + 1)^{-1} M^{1-\alpha+\epsilon}. \end{aligned}$$

Finally, letting M to infinity we obtain

$$\lim_{t \rightarrow 0} \psi^-(1/t) I_1 = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \text{Per}(\Omega) \int_0^{\infty} s^d p_1^{(\alpha)}(se_d) ds.$$

It is known that, see e.g. [1, Lemma 4.1],

$$\int_0^{\infty} s^d p_1^{(\alpha)}(se_d) ds = \pi^{-(d+1)/2} \Gamma((d+1)/2) \Gamma(1 - 1/\alpha)$$

and we conclude the result. \square

3.3. Proof of Theorem 3.

Proof of Theorem 3. We consider two cases: the first is $\gamma_0 = 0$. Then $X_t = X_t^0$, where \mathbf{X}^0 is as in Lemma 2 and we have

$$(31) \quad t^{-1}H(t) = \int_{\Omega} \frac{1 - \mathbb{P}(X_t + x \in \Omega)}{t} dx = t^{-1}(g_{\Omega}(0) - P_t g_{\Omega}(0)),$$

which converges to $-\mathcal{L}g_{\Omega}(0) = \text{Per}_{\mathbf{X}}(\Omega)$ according to Subsection 2.3 and Lemma 2.

In the case $\gamma_0 \neq 0$, we write $X_t = X_t^0 + t\gamma_0$, where \mathbf{X}^0 is again as in Lemma 2, and thus

$$\begin{aligned} t^{-1}H(t) &= \int_{\Omega} \frac{1 - \mathbb{P}(X_t^0 + t\gamma_0 + x \in \Omega)}{t} dx \\ &= \int_{\Omega} \frac{1 - \mathbb{P}(X_t^0 + x \in \Omega)}{t} dx + \int_{\Omega} \frac{\mathbb{P}(X_t^0 + x \in \Omega) - \mathbb{P}(X_t^0 + t\gamma_0 + x \in \Omega)}{t} dx. \end{aligned}$$

By (31) we obtain

$$(32) \quad \lim_{t \rightarrow 0} \int_{\Omega} \frac{1 - \mathbb{P}(X_t^0 + x \in \Omega)}{t} dx = \text{Per}_{\mathbf{X}}(\Omega).$$

We denote by $p_t^0(dx)$ and h^0 the transition probabilities and the function introduced in (6), respectively, corresponding to the process \mathbf{X}^0 . For the second integral we proceed as follows

$$\begin{aligned} \int_{\Omega} \frac{\mathbb{P}(X_t^0 + x \in \Omega) - \mathbb{P}(X_t^0 + t\gamma_0 + x \in \Omega)}{t} dx \\ = \int_{\mathbb{R}^d} \mathbf{1}_{\Omega}(x) t^{-1} \int_{\mathbb{R}^d} (\mathbf{1}_{\Omega}(y + x) - \mathbf{1}_{\Omega}(y + x + t\gamma_0)) p_t^0(dy) dx \\ = \int_{\mathbb{R}^d} \frac{g_{\Omega}(y) - g_{\Omega}(y + t\gamma_0)}{t} p_t^0(dy). \end{aligned}$$

We take $\epsilon > 0$. Using (iv) of Proposition 1 and (16) we write

$$\begin{aligned} (33) \quad & \left| \int_{\|y\| > \epsilon t} \frac{g_{\Omega}(y) - g_{\Omega}(y + t\gamma_0)}{t} p_t^0(dy) \right| \leq \|\gamma_0\| \mathbb{P}(\|X_t^0\| > \epsilon t) \leq C\|\gamma_0\| t h^0(\epsilon t) \\ & = C\|\gamma_0\| t \int_{\mathbb{R}^d} \left(1 \wedge \frac{\|y\|^2}{(\epsilon t)^2}\right) \nu(dy) = C\|\gamma_0\| t \int_{\mathbb{R}^d} \left(1 \wedge \frac{\|y\|}{\epsilon t}\right)^2 \nu(dy) \\ & \leq C\|\gamma_0\| t \int_{\mathbb{R}^d} \left(1 \wedge \frac{\|y\|}{\epsilon t}\right) \nu(dy) = C\|\gamma_0\| \int_{\mathbb{R}^d} \left(t \wedge \frac{\|y\|}{\epsilon}\right) \nu(dy). \end{aligned}$$

By the Lebesgue dominated convergence theorem the last quantity tends to zero as t goes to zero. For the other part of the integral we have

$$\begin{aligned} \int_{\|y\| \leq \epsilon t} \frac{g_{\Omega}(y) - g_{\Omega}(y + t\gamma_0)}{t} p_t^0(dy) &= \int_{\|y\| \leq \epsilon t} \frac{g_{\Omega}(y) - g_{\Omega}(0)}{t} p_t^0(dy) \\ &+ \int_{\|y\| \leq \epsilon t} \frac{g_{\Omega}(0) - g_{\Omega}(t\gamma_0)}{t} p_t^0(dy) + \int_{\|y\| \leq \epsilon t} \frac{g_{\Omega}(\gamma_0 t) - g_{\Omega}(y + t\gamma_0)}{t} p_t^0(dy) = I_1 + I_2 + I_3. \end{aligned}$$

Handling with I_1 is easy in front of condition (iv) of Proposition 1. Indeed,

$$|I_1| \leq \int_{\|y\| \leq \epsilon t} \frac{|g_{\Omega}(y) - g_{\Omega}(0)|}{\|y\|} \cdot \frac{\|y\|}{t} p_t^0(dy) \leq L\epsilon \int_{\|y\| \leq \epsilon t} p_t^0(dy) \leq L\epsilon.$$

Similarly we estimate I_3 . The integral I_2 equals

$$I_2 = \frac{g_\Omega(0) - g_\Omega\left((\|\gamma_0\|t)^{\frac{\gamma_0}{\|\gamma_0\|}}\right)}{\|\gamma_0\|t} \|\gamma_0\| \int_{\|y\| \leq \epsilon t} p_t^0(dy).$$

Using (13) we obtain

$$(34) \quad \lim_{t \rightarrow 0} \frac{g_\Omega(0) - g_\Omega\left((\|\gamma_0\|t)^{\frac{\gamma_0}{\|\gamma_0\|}}\right)}{\|\gamma_0\|t} = \frac{V_{\frac{\gamma_0}{\|\gamma_0\|}}(\Omega)}{2}.$$

Moreover, we claim that

$$\lim_{t \rightarrow 0} \int_{\|y\| \leq \epsilon t} p_t^0(dy) = 1.$$

Indeed, we have

$$\int_{\|y\| \leq \epsilon t} p_t^0(dy) = 1 - \mathbb{P}(\|X_t^0\| > \epsilon t).$$

Proceeding in the same fashion as in (33) we show that $\mathbb{P}(\|X_t^0\| > \epsilon t)$ tends to zero as t goes to zero, which gives the claim. Finally, equations (32) and (34) imply the result. \square

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